

BRUHAT-CHEVALLEY ORDER ON THE ROOK MONOID

MAHIR BILEN CAN,
LEX E. RENNER

ABSTRACT. The rook monoid R_n is the finite monoid whose elements are the $0 - 1$ matrices with at most one nonzero entry in each row and column. The group of invertible elements of R_n is isomorphic to the symmetric group S_n . The natural extension to R_n of the Bruhat-Chevalley ordering on the symmetric group is defined in [4]. In this paper, we find an efficient, combinatorial description of the Bruhat-Chevalley ordering on R_n . We also give a useful, combinatorial formula for the length function on R_n .

1. INTRODUCTION

Let GL_n be the general linear group over an algebraically closed field \mathbb{F} . There is a much-studied decomposition of GL_n into double cosets of the Borel subgroup $B \subset GL_n$ of invertible upper triangular matrices

$$(1.1) \quad GL_n = \bigcup_{w \in S_n} BwB,$$

where the union is indexed by the symmetric group S_n . Elements of S_n are identified with $0 - 1$ matrices with exactly one nonzero entry in each row and column.

The decomposition in (1.1) is often referred to as the Bruhat decomposition and it holds, more generally, for reductive groups and reductive monoids (see [2, 4]). In the case of the monoid M_n of $n \times n$ matrices, the Bruhat decomposition is given by

$$(1.2) \quad M_n = \bigcup_{\sigma \in R_n} B\sigma B,$$

where the union is indexed by the rook monoid R_n . The elements of R_n are identified with $0 - 1$ matrices which have at most one nonzero entry in each row and column.

The Bruhat-Chevalley order on S_n is defined in terms of the inclusion relationships between double cosets in (1.1). Namely, if $v, w \in S_n$, then

$$(1.3) \quad v \leq w \iff BvB \subseteq \overline{BwB},$$

where the overline stands for the Zariski closure in GL_n .

There is a natural extension of this partial order on the rook monoid R_n (see [2, 4] for more details).

$$(1.4) \quad \sigma \leq \tau \iff B\sigma B \subseteq \overline{B\tau B},$$

for $\sigma, \tau \in R_n$.

In [3], Putcha describes the partial ordering (1.4) for the constant-rank subsets of the rook monoid in terms of the Bruhat order of related symmetric groups (he describes this partial order, much more generally, for any J -class of a Renner monoid).

In [1], using a partial ordering exactly like (1.4), Miller and Sturmfels study the poset of Zariski closures of $B \times B_+$ -orbits on the space of the $k \times l$ matrices. Here B denotes the group of the invertible upper triangular $k \times k$ matrices, and B_+ denotes the group of invertible lower triangular $l \times l$ matrices. These $B \times B_+$ -orbits are indexed by the $0-1$, $k \times l$ matrices with at most one nonzero entry in each row and column.

For computational purposes, one would like to have an efficient, combinatorial characterization of the Bruhat-Chevalley ordering on R_n . This characterization, in the case of the symmetric group, had been explained to us by V. Deodhar.

1.0.1. Deodhar's characterization. For an integer valued vector $a = (a_1, \dots, a_n) \in \mathbb{Z}^n$, let $\tilde{a} = (a_{\alpha_1}, \dots, a_{\alpha_n})$ be the rearrangement of the entries a_1, \dots, a_n of a in a non-increasing fashion;

$$a_{\alpha_1} \geq a_{\alpha_2} \geq \dots \geq a_{\alpha_n}.$$

The *containment ordering*, “ \leq_c ,” on \mathbb{Z}^n is then defined by

$$a = (a_1, \dots, a_n) \leq_c b = (b_1, \dots, b_n) \iff a_{\alpha_j} \leq b_{\alpha_j} \text{ for all } j = 1, \dots, n.$$

where $\tilde{a} = (a_{\alpha_1}, \dots, a_{\alpha_n})$, and $\tilde{b} = (b_{\alpha_1}, \dots, b_{\alpha_n})$.

Let $k \in \{1, \dots, n\}$. The k 'th *truncation*, $a(k)$ of $a = (a_1, \dots, a_n)$ is defined to be

$$a(k) = (a_1, a_2, \dots, a_k).$$

We represent the elements of the symmetric group S_n by n -tuples; for $v \in S_n$ let (v_1, \dots, v_n) be the sequence where v_j is the row index of the nonzero entry in the j 'th column of the matrix v . For example, the 4-tuple associated with the permutation matrix

$$(1.5) \quad v = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \text{ is } (3142).$$

In general, we write $v = (v_1, \dots, v_n)$ for the corresponding permutation matrix.

Definition 1.1. The Deodhar ordering, \leq_D , on S_n is defined by

$$(1.6) \quad v = (v_1, \dots, v_n) \leq_D w = (w_1, \dots, w_n) \iff \widetilde{v(k)} \leq_c \widetilde{w(k)} \text{ for all } k = 1, \dots, n.$$

Remark 1.2. The Deodhar ordering, \leq_D is equivalent to the Bruhat-Chevalley ordering on S_n . Although there seems to be no published proof of this fact, it follows as a corollary of our main theorem.

For the rook monoid R_n , a combinatorial description of the Bruhat-Chevalley ordering is given in [2]. We summarize it here.

We represent the elements of R_n by n -tuples of nonnegative integers. Given $x = (x_{ij}) \in R_n$, let (a_1, \dots, a_n) be the sequence defined by

$$(1.7) \quad a_j = \begin{cases} 0, & \text{if the } j\text{th column consists of zeros;} \\ i, & \text{if } x_{ij} = 1. \end{cases}$$

For example, the sequence associated with the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

is (3040).

Theorem 1.3. [2] *Let $x = (a_1, \dots, a_n)$, $y = (b_1, \dots, b_n) \in R_n$. Then the Bruhat-Chevalley order on R_n is the smallest partial order on R_n generated by declaring $x \leq y$ if either*

- (1) *there exists an $1 \leq i \leq n$ such that $b_i > a_i$ and $b_j = a_j$ for all $j \neq i$, or*
- (2) *there exist $1 \leq i < j \leq n$ such that $b_i = a_j$, $b_j = a_i$ with $b_i > b_j$, and for all $k \notin \{i, j\}$, $b_k = a_k$.*

For example, let $x = (21403)$ and $y = (35201)$ in R_5 . Then $x \leq_{PPR} y$ because

$$\begin{aligned} (21403) &\leq_{PPR} (31402) \text{ by Theorem 1.3 part 2} \\ &\leq_{PPR} (34102) \text{ by Theorem 1.3 part 2} \\ &\leq_{PPR} (35102) \text{ by Theorem 1.3 part 1} \\ &\leq_{PPR} (35201) \text{ by Theorem 1.3 part 2.} \end{aligned}$$

Remark 1.4. In Proposition 15.23 of [1], Miller and Sturmfels describe the particular case of Theorem 1.3 where $y \in S_n$.

For the sake of notation, the partial ordering defined by the Theorem 1.3 is denoted by “ \leq_{PPR} ,” and referred to as the “Pennell-Putcha-Renner” ordering on R_n .

Notice that Deodhar’s ordering (1.6) on S_n can be defined verbatim on the rook monoid.

Definition 1.5. The Deodhar ordering \leq_D on R_n is defined as follows.

$$(1.8) \quad v = (v_1, \dots, v_n) \leq_D w = (w_1, \dots, w_n) \iff \widetilde{v(k)} \leq_c \widetilde{w(k)} \text{ for all } k = 1, \dots, n.$$

Example 1.6. Let $x = (4, 0, 2, 3, 1)$, and let $y = (4, 3, 0, 5, 1)$. Then $x \leq_D y$, because

$$\begin{aligned} \widetilde{x(1)} &= (4) \leq_c \widetilde{y(1)} = (4), \\ \widetilde{x(2)} &= (4, 0) \leq_c \widetilde{y(2)} = (4, 3), \\ \widetilde{x(3)} &= (4, 2, 0) \leq_c \widetilde{y(3)} = (4, 3, 0), \\ \widetilde{x(4)} &= (4, 3, 2, 0) \leq_c \widetilde{y(4)} = (5, 4, 3, 0), \\ \widetilde{x(5)} &= (4, 3, 2, 1, 0) \leq_c \widetilde{y(5)} = (5, 4, 3, 1, 0). \end{aligned}$$

The main theorem of this article is that, on R_n , the Deodhar ordering and the Pennell-Putcha-Renner ordering are identical.

The organization of the paper is as follows. In Section 2, we study the length function on R_n . We show that

Theorem 1.7. *Let $x = (a_1, \dots, a_n) \in R_n$. Then, the dimension $\ell(x) = \dim(BxB)$ of the orbit BxB , is given by*

$$(1.9) \quad \ell(x) = \left(\sum_{i=1}^n a_i^* \right) - \text{coinv}(x), \text{ where } a_i^* = \begin{cases} a_i + n - i, & \text{if } a_i \neq 0, \\ 0, & \text{if } a_i = 0. \end{cases}$$

In Section 3, we prove two lemmas, which sharpen the theorem of Pennell, Putcha and Renner. In Section 4, we find an equivalent description of the Deodhar's ordering. Finally, in Section 5, we prove that

Theorem 1.8. *The Deodhar ordering \leq_D on R_n is the same as the Pennell-Putcha-Renner \leq_{PPR} ordering on R_n .*

2. THE LENGTH FUNCTION.

It is well known that the symmetric group S_n is a graded poset, grading given by the length function

$$(2.1) \quad \ell(w) = \dim(BwB) = \text{inv}(w) + \dim(B) = \text{inv}(w) + \binom{n+1}{2},$$

where $w \in S_n$, and

$$(2.2) \quad \text{inv}(w) = |\{(i, j) : 1 \leq i < j \leq n, w_i > w_j\}|.$$

In [4], it is shown that the rook monoid is a graded poset, with respect to the length function

$$(2.3) \quad \ell(\sigma) = \dim(B\sigma B), \sigma \in R_n.$$

In this section we give a combinatorial formula, similar to (2.1), for the length function on R_n .

Let R_n^1 be the set of all rank one elements of R_n . We denote the elements of R_n^1 by $E_{ij} = (e_{rs}) \in R_n$, where

$$e_{rs} = \begin{cases} 1, & \text{if } r = i, \text{ and } s = j, \\ 0, & \text{otherwise.} \end{cases}$$

Let \mathbf{T}_n be the set of all upper triangular matrices in \mathbf{M}_n .

Lemma 2.1. *Let B be the Borel subgroup of invertible upper triangular matrices, and let $x = (x_{rs})$ be an element of R_n . Then, the dimension $\dim(Bx)$ is equal to the dimension of the linear subspace $\mathbf{T}_n x$ of \mathbf{M}_n , which is spanned by the following set;*

$$\{E_{ij} \in R_n^1 : \text{there exists a nonzero entry } x_{rs} \text{ of } x \text{ with } s = j \text{ and } r \geq i\}.$$

Proof. The linearity of $\mathbf{T}_n x \subset \mathbf{M}_n$ is clear. Since $\overline{Bx} = \overline{B}x = \mathbf{T}_n x$, and since the geometric dimension of a linear space is the same as its vector space dimension, $\dim(Bx) = \dim(\overline{Bx}) = \dim(\mathbf{T}_n x)$. It is easy to see that, $\mathbf{T}_n x$ is spanned by $R_n^1 \cap \mathbf{T}_n x$. Matrix multiplication shows that $E_{ij} \in R_n^1 \cap \mathbf{T}_n x$ if and only if there exists a nonzero entry x_{rs} of x with $r \geq i$ and $s = j$. \square

Lemma 2.2. *Let B be the Borel subgroup of invertible upper triangular matrices, and let $x = (x_{rs})$ be an element of R_n . Then, the dimension $\dim(xB)$ is equal to the dimension of the linear subspace $x\mathbf{T}_n$ of \mathbf{M}_n , which is spanned by the following set;*

$$\{E_{ij} \in R_n^1 : \text{there exists a nonzero entry } x_{rs} \text{ of } x \text{ with } r = i \text{ and } s \leq j\}.$$

Proof. Identical to the proof of Lemma 2.1. \square

Example 2.3. Let $x \in R_4$ be given by the matrix

$$x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Then, a generic element of $\mathbf{T}_4 x$ is of the form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} a_{14} & 0 & a_{12} & a_{13} \\ a_{24} & 0 & a_{22} & a_{23} \\ a_{34} & 0 & 0 & a_{33} \\ a_{44} & 0 & 0 & 0 \end{pmatrix},$$

for some $a_{ij} \in \mathbb{F}$. Therefore, $\dim(\mathbf{T}_4 x) = 9$. Similarly, an arbitrary element of $x\mathbf{T}_4$ is of the form

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ 0 & b_{22} & b_{23} & b_{24} \\ 0 & 0 & b_{33} & b_{34} \\ 0 & 0 & 0 & b_{44} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & b_{33} & b_{34} \\ 0 & 0 & 0 & b_{44} \\ b_{11} & b_{12} & b_{13} & b_{14} \end{pmatrix},$$

for some $b_{ij} \in \mathbb{F}$. Thus $\dim(x\mathbf{T}_4) = 7$.

Remark 2.4. Let $x = (a_1, \dots, a_n)$ be the “one line” representation for $x = (x_{rs}) \in R_n$, as in 1.7. If $a_i \neq 0$ for some $i \in \{1, \dots, n\}$, then a_i is the row index of a nonzero entry $x_{a_i i}$ of x . Therefore, $E_{r,s} \in R_n^1 \cap \mathbf{T}_n x$ if and only if there exists a nonzero entry of x at the position (a_i, i) with $s = i$ and $r \geq a_i$. Similarly, $E_{r,s} \in R_n^1 \cap x\mathbf{T}_n$ if and only if there exists a nonzero entry of x at the position (a_j, j) with $r = a_j$ and $s \leq j$.

Definition 2.5. Let $x = (a_1, \dots, a_n) \in R_n$. A pair (i, j) of indices $1 \leq i < j \leq n$ is called a *coinversion pair* for x , if $0 < a_i < a_j$. By abuse of notation, we use *coinv* for both the set of coinversion pairs of x , as well as its cardinality.

Example 2.6. Let $x = (4, 0, 2, 3)$. Then, the only coinversion pair for x is $(3, 4)$. Therefore, $\text{coinv}(x) = 1$.

Theorem 2.7. Let $x = (a_1, \dots, a_n) \in R_n$. Then, the dimension, $\ell(x) = \dim(BxB)$ of the orbit BxB is given by

$$(2.4) \quad \ell(x) = \left(\sum_{i=1}^n a_i^* \right) - \text{coinv}(x), \text{ where } a_i^* = \begin{cases} a_i + n - i, & \text{if } a_i \neq 0 \\ 0, & \text{if } a_i = 0 \end{cases}$$

Proof. Recall from [5] that the dimension of the orbit BxB can be calculated by

$$(2.5) \quad \dim(BxB) = \dim(Bx) + \dim(xB) - \dim(Bx \cap xB).$$

By Lemma 2.1, $\dim(Bx)$ is the number of positions on or above some nonzero entry of the matrix $x \in R_n$. In other words, by the Remark 2.4, if $x = (a_1, \dots, a_n)$, then $\sum_{i=1}^n a_i$ is equal to $\dim(Bx)$.

Similarly, by Lemma 2.2, $\dim(xB)$ is the number of positions on or to the right of some nonzero entry of x . The number of positions on and to the right of the nonzero entry at the (a_i, i) 'th position of the matrix x is equal to $n - i + 1$. This shows that

$$\dim(Bx) + \dim(xB) = \sum_{i=1}^n \overline{a_i},$$

where

$$\overline{a_i} = \begin{cases} a_i + n - i + 1, & \text{if } a_i \neq 0, \\ 0, & \text{if } a_i = 0. \end{cases}$$

The number of nonzero entries of x is denoted by $\text{rank}(x)$. Thus, we have

$$\dim(Bx) + \dim(xB) = \sum_{i=1}^n a_i^* + \text{rank}(x),$$

where

$$a_i^* = \begin{cases} a_i + n - i, & \text{if } a_i \neq 0, \\ 0, & \text{if } a_i = 0. \end{cases}$$

Therefore, it is enough to prove that

$$\dim(Bx \cap xB) = \text{rank}(x) + \text{coinv}((a_1, \dots, a_n)).$$

By a similar argument as in the proof of Lemma 2.1, the dimension of $Bx \cap xB$ is equal to $\dim(\mathbf{T}_n x \cap x \mathbf{T}_n)$, which is equal to the cardinality of the set $R_n^1 \cap \mathbf{T}_n x \cap x \mathbf{T}_n$.

Let $E_{rs} \in R_n^1 \cap \mathbf{T}_n x \cap x \mathbf{T}_n$ be a rank 1 element whose nonzero entry is at the (r, s) 'th position. By the Remark 2.4, $E_{rs} \in R_n^1 \cap \mathbf{T}_n x \cap x \mathbf{T}_n$ if and only if there exist nonzero entries of x at the positions (a_i, i) and (a_j, j) such that $r \geq a_i$, $s = i$ and $r = a_j$, $s \leq j$. We have two possibilities. Either $(a_i, i) = (a_j, j)$, or not. Clearly, the number of times that the equality $(a_i, i) = (a_j, j)$ holds true is equal to the $\text{rank}(x)$. On the other hand, if $(a_i, i) \neq (a_j, j)$, then we see that $i < j$ and $0 < a_i < a_j$. Therefore, the number of times that $(a_i, i) \neq (a_j, j)$, is equal to the number of coinversions of the sequence (a_1, \dots, a_n) . Therefore,

$$\dim(Bx \cap xB) = |R_n^1 \cap \mathbf{T}_n x \cap x \mathbf{T}_n| = \text{rank}(x) + \text{coinv}((a_1, \dots, a_n)).$$

□

Remark 2.8. Let $x = (a_1, \dots, a_n) \in R_n$ be a permutation. Then

$$\begin{aligned} \ell(x) &= \left(\sum_{i=1}^n a_i + n - i \right) - \text{coinv}(x) \\ &= \binom{n+1}{2} + \binom{n}{2} - \text{coinv}(x) \\ &= \binom{n+1}{2} + \text{inv}(x), \end{aligned}$$

which agrees with the formula (2.1).

Example 2.9. We continue with the notation of the example 2.3. The generic element of $\mathbf{T}_4x \cap x\mathbf{T}_4$ has the form

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \\ * & 0 & 0 & 0 \end{pmatrix},$$

where $*$ denotes an arbitrary element of \mathbb{F} . Therefore, $\dim(\mathbf{T}_4x \cap x\mathbf{T}_4) = 4$, and by formula 2.5, we have $\dim(BxB) = 9 + 7 - 4 = 12$. On the other hand, x is represented in “one line” notation by $(4, 0, 2, 3)$, and by Theorem 1.7 we have

$$\ell(x) = (4 + 4 - 1) + (2 + 4 - 3) + (3 + 4 - 4) - 1 = 12.$$

3. TWO IMPORTANT LEMMAS.

Recall that we denote the Bruhat-Chevalley ordering on R_n , as in Theorem 1.3, by \leq_{PPR} . The following two lemmas are critical for deciding if $x \leq_{PPR} y$ is a covering relation.

Lemma 3.1. *Let $x = (a_1, \dots, a_n)$ and $y = (b_1, \dots, b_n)$ be elements of R_n . Suppose that $a_k = b_k$ for all $k = \{1, \dots, \widehat{i}, \dots, n\}$ and $a_i < b_i$. Then, $\ell(y) = \ell(x) + 1$ if and only if either*

- (1) $b_i = a_i + 1$, or
- (2) *there exists a sequence of indices $1 \leq j_1 < \dots < j_s < i$ such that the set $\{a_{j_1}, \dots, a_{j_s}\}$ is equal to $\{a_i + 1, \dots, a_i + s\}$, and $b_i = a_i + s + 1$.*

Proof. Note that by the hypotheses of the lemma, Theorem 1.3 implies that $x \leq_{PPR} y$. We first show that if (1) or (2) holds, then $\ell(y) = \ell(x) + 1$, in other words y covers x .

If $b_i = a_i + 1$, then by the Theorem 2.7 the lemma follows. So, we assume that there exists a sequence of indices $1 \leq j_1 < \dots < j_s < i$ such that the set $\{a_{j_1}, \dots, a_{j_s}\}$ is equal to $\{a_i + 1, \dots, a_i + s\}$, and $b_i = a_i + s + 1$. Then,

$$\begin{aligned} \ell(y) &= \sum_{j=1}^n b_j^* - \text{coinv}(y) \\ &= \left(\sum_{j=1, j \neq i}^n a_j^* \right) + b_i^* - \text{coinv}(y) \\ &= \left(\sum_{j=1, j \neq i}^n a_j^* \right) + a_i + s + 1 + n - i - \text{coinv}(y) \\ &= \left(\sum_{j=1}^n a_j^* \right) + s + 1 - \text{coinv}(y). \end{aligned}$$

Now it suffices to show that $\text{coinv}(y) = s + \text{coinv}(x)$. Observe that, when we replace a_i by b_i , the following set of pairs, which are not coinversion pairs for x ,

$$\{(j_k, i) \mid k = 1, \dots, s\},$$

become coinversion pairs for y . Also, upon replacing the entry a_i by b_i , a coinversion pair of x of the form (l, i) or (i, l) (where $l \neq j_k$) stays to be a coinversion pair for y . Therefore,

$$\text{coinv}(y) = s + \text{coinv}(x),$$

and hence $\ell(y) = \ell(x) + 1$.

We proceed to prove the converse statement. Assume that $\ell(y) = \ell(x) + 1$. Since $b_i > a_i$, there exists $d > 0$ such that $b_i = a_i + d$. Without loss of generality we may assume that $d > 1$. Then the length of y can be computed as follows.

$$\begin{aligned} \ell(y) &= \sum_{j=1}^n b_j^* - \text{coinv}(y) \\ &= \left(\sum_{j=1, j \neq i}^n a_j^* \right) + b_i^* - \text{coinv}(y) \\ &= \left(\sum_{j=1, j \neq i}^n a_j^* \right) + a_i + d + n - i - \text{coinv}(y) \\ &= \left(\sum_{j=1}^n a_j^* \right) + d - \text{coinv}(y) \\ &= \ell(x) + d + \text{coinv}(x) - \text{coinv}(y). \end{aligned}$$

Hence $d + \text{coinv}(x) - \text{coinv}(y) = 1$, or $\text{coinv}(y) - \text{coinv}(x) = d - 1$. We inspect the difference $\text{coinv}(x) - \text{coinv}(y)$ more closely. If (k, i) with $k < i$ is a coinversion for x , then it stays to be a coinversion for y , as well. Clearly this is also true for the pairs of the form (k, l) where $k < i < l$, or $i < k < l$, or $k < l < i$.

Therefore, the difference between $\text{coinv}(y)$ and $\text{coinv}(x)$ occurs at the pairs of the form

- (1) (k, i) , $k < i$ such that $a_i < a_k < b_i$, or
- (2) (i, l) , $i < l$, such that $a_i < a_l < b_i$.

In the first case, some new coinversions are added, and in the second case some coinversions are deleted. Let us call the number of pairs of the first type by n_1 and the number of pairs of the second type by n_2 . Then, $\text{coinv}(y) = \text{coinv}(x) + n_1 - n_2$, or $\text{coinv}(y) - \text{coinv}(x) = n_1 - n_2$. Obviously $0 \leq n_1, n_2 \leq d-1$ (because $b_i = a_i + d$). Hence, we have that $n_1 = d-1$, and that $n_2 = 0$. Therefore, the following is true: any a_k between a_i and $a_i + d = b_i$ appears before the i 'th position. This completes the proof. \square

Example 3.2. Let $x = (4, 0, 5, 0, 3, 1)$, and let $y = (4, 0, 5, 0, 6, 1)$. Then $\ell(x) = 21$, and $\ell(y) = 22$. Let $z = (6, 0, 5, 0, 3, 1)$. Then $\ell(z) = 23$.

Lemma 3.3. Let $x = (a_1, \dots, a_n)$ and $y = (b_1, \dots, b_n)$ be two elements of R_n . Suppose that $a_j = b_i$, $a_i = b_j$ and $b_j < b_i$ where $i < j$. Furthermore, suppose that for all $k \in \{1, \dots, \widehat{i}, \dots, \widehat{j}, \dots, n\}$, $a_k = b_k$. Then, $\ell(y) = \ell(x) + 1$ if and only if for $s = i + 1, \dots, j - 1$, either $a_j < a_s$, or $a_s < a_i$.

Proof. Suppose that x and y are as in the hypothesis. Also suppose also that $\ell(y) = \ell(x) + 1$. We proceed to show that for $s = i + 1, \dots, j - 1$, either $a_j < a_s$, or $a_s < a_i$. Clearly, the sets $\{a_1, \dots, a_n\}$ and $\{b_1, \dots, b_n\}$ are equal, hence $\sum_{t=1}^n a_t = \sum_{t=1}^n b_t$. Therefore, the difference between $\ell(x)$ and $\ell(y)$ is determined by the associated coinversion sets of x and y .

Assume that there exists an $s \in \{i + 1, \dots, j - 1\}$ such that $a_i < a_s < a_j$. Then, upon interchanging a_i with a_j to get y from x , the pairs (i, s) , (s, j) and (i, j) are no longer coinversions for y . This shows that for every $s = i + 1, \dots, j - 2$ with $a_i < a_s < a_j$, we obtain that $\ell(y) \geq \ell(x) + 2$. This contradicts the assumption that $\ell(y) = \ell(x) + 1$. Therefore, there exists no $s \in \{i + 1, \dots, j - 1\}$ such that $a_i < a_s < a_j$.

Conversely, assume that for every $s = i + 1, \dots, j - 1$, we have $a_i > a_s$ or $a_s > a_j$. If $a_i > a_s$, then, the pair (s, j) is a coinversion pair for both x and y . On the other hand, the pair (i, s) is neither a coinversion for x nor for y . Similarly, if $(a_s > a_j)$, then the pair (i, s) is a coinversion pair for both x and y . Also, the pair (s, j) is not a coinversion pair for x and neither for y . Therefore, we conclude that at any pair of the form (k, l) with $i \leq k < l \leq j$, the coinversion is not affected. It remains to check pairs of the form (k, l) with either $k < i$, or $j < k$. In the first case, i.e., $k < i$, as a_i is interchanged with a_j , the contribution of (k, l) to the coinversion situation does not change, since relative positions of a_k and a_l do not alter. Similarly, in the second case, i.e., $j < k$, since the relative positions of a_k and a_l do not alter, their contribution to coinversion do not change. Therefore, the only coinversion change occurs at the pair (i, j) , and hence, $\ell(y) = \ell(x) + 1$. This completes the proof. \square

Example 3.4. Let $x = (2, 6, 5, 0, 4, 1, 7)$, and let $y = (4, 6, 5, 0, 2, 1, 7)$. Then $\ell(x) = 35$, and $\ell(y) = 36$. Let $z = (7, 6, 5, 0, 4, 1, 2)$. Then $\ell(z) = 42$.

4. ANOTHER CHARACTERIZATION OF \leq_D .

As mentioned in the introduction, our goal is to show that the \leq_D ordering on R_n is the same as to the \leq_{PPR} ordering. In this section, we find another, useful characterization of the Deodhar ordering.

Definition 4.1. Let $x = (a_1, \dots, a_n) \in R_n$, and let $r \in \{1, \dots, n\}$, and finally let $a \in \mathbb{Z}$. We define

$$\Gamma(x, a) = \{a_i \in x \mid a_i > a\}.$$

Remark 4.2. Let a_i be a nonzero entry of $x = (a_1, \dots, a_n) \in R_n$. Then, $|\Gamma(x, a_i)| + 1$ is the position of a_i in the reordering $\tilde{x} = (a_{\alpha_1} \geq \dots \geq a_{\alpha_n})$ of the entries of x . For example, if $x = (3, 0, 5, 1, 0, 4)$, then $\tilde{x} = (5, 4, 3, 1, 0, 0)$, and $|\Gamma(x, 1)| + 1 = 4$.

Proposition 4.3. Let $x = (a_1, \dots, a_n)$ and $y = (b_1, \dots, b_n)$ be two elements from R_n . Then $x \leq_c y$ if and only if $|\Gamma(x, a_k)| \leq |\Gamma(y, a_k)|$ for all $k = 1, \dots, n$.

Proof. Let $\tilde{y} = (b_{\alpha_1} \geq \dots \geq b_{\alpha_n})$ and $\tilde{x} = (a_{\alpha_1} \geq \dots \geq a_{\alpha_n})$ be the reorderings of the entries of y and of x respectively. Then, by the Remark 4.2, $a_{\alpha_{s+1}}$ is the entry a_k of x for which $|\Gamma(x, a_k)| = s$. Therefore, $b_{\alpha_{s+1}} \geq a_{\alpha_{s+1}}$ if and only if the number of entries of y which are larger than a_k is more than the number of entries of x which are larger than a_k . In other words, $b_{\alpha_{s+1}} \geq a_{\alpha_{s+1}}$ if and only if $|\Gamma(x, a_k)| \leq |\Gamma(y, a_k)|$. Thus $x \leq_c y$ if and only if $|\Gamma(x, a_k)| \leq |\Gamma(y, a_k)|$, for all $k = 1, \dots, n$. \square

As a corollary of the Proposition 4.3, we have

Corollary 4.4. Let $x = (a_1, \dots, a_n)$, and $y = (b_1, \dots, b_n)$ be two elements of R_n . Then $y \geq_D x$ if and only if for all $1 \leq k \leq n$ and for all $m \leq k$, $|\Gamma(x(k), a_m)| \leq |\Gamma(y(k), a_m)|$.

Proof. Immediate from Proposition 4.3, and the definition of the Deodhar ordering. \square

Example 4.5. Let $x = (a_1, a_2, a_3) = (1, 0, 3)$ and let $y = (b_1, b_2, b_3) = (3, 0, 2)$. Then

$$\begin{aligned} |\Gamma(x(1), a_1)| = 0 &\leq |\Gamma(y(1), a_1)| = 1, \\ |\Gamma(x(2), a_1)| = 0 &\leq |\Gamma(y(2), a_1)| = 1, \\ |\Gamma(x(2), a_2)| = 1 &\leq |\Gamma(y(2), a_2)| = 2, \\ |\Gamma(x(3), a_1)| = 1 &\leq |\Gamma(y(3), a_1)| = 2, \\ |\Gamma(x(3), a_2)| = 2 &\leq |\Gamma(y(3), a_2)| = 2, \\ |\Gamma(x(3), a_3)| = 0 &\leq |\Gamma(y(3), a_3)| = 0. \end{aligned}$$

Therefore, $x \leq_D y$.

Remark 4.6. It follows from the definition of the Deodhar ordering that if $(a_1, \dots, a_n) \leq_D (b_1, \dots, b_n)$, then $(a_1, \dots, a_k) \leq_D (b_1, \dots, b_k)$ for any $k \in \{1, \dots, n\}$. Also, by repeated application of Proposition 4.3, it follows that

$$(a_1, \dots, a_k, c_{k+1}, \dots, c_m) \leq_D (b_1, \dots, b_k, c_{k+1}, \dots, c_m)$$

for any set $\{c_{k+1}, \dots, c_m\}$ of nonnegative integers.

5. THE MAIN THEOREM.

We show in this section that the covering relation for the ordering \leq_{PPR} on R_n is the same as the covering relation for the ordering \leq_D on R_n . Our notation for these covering relations is “ $y \rightarrow_D x$,” and “ $y \rightarrow_{PPR} x$,” respectively.

Lemma 5.1. *Let $x = (a_1, \dots, a_n)$, $y = (b_1, \dots, b_n)$ and $z = (c_1, \dots, c_n)$ be three elements from R_n such that $a_k = b_k$ for all $k \in \{1, \dots, \widehat{i}, \dots, n\}$ and $a_i < b_i$. Furthermore, suppose that $c_k = a_k$ for $k = 1, \dots, i$. If $x \leq_D z \leq_D y$ and $\ell(y) = \ell(x) + 1$, then $z = x$.*

Proof. Assume otherwise that $z \neq x$. Let $j > i$ be the smallest number such that $c_k = a_k$ for $k < j$ but $c_j \neq a_j$. Since $x \leq_D z$, it cannot be true that $c_j < a_j$. So, we have that $a_j < c_j$. This, in particular, implies that c_j is nonzero.

We now compare c_j with a_i . Observe that $c_j = a_i$ is not possible. Thus, there are two cases; either $c_j < a_i$ or $a_i < c_j$.

We proceed with the first case. Then, we have $a_j = b_j < c_j < a_i = c_i < b_i$. Recall that $\Gamma(z(j), b_j) = \{c_k \mid b_j < c_k, k = 1, \dots, j\}$, and that $\Gamma(y(j), b_j) = \{b_k \mid b_j < b_k, k = 1, \dots, j\}$.

Since,

$$\{b_1, \dots, b_j\} \setminus \{b_i, b_j\} = \{c_1, \dots, c_j\} \setminus \{c_j, c_i\}.$$

and since $b_j < c_j < c_i$, we see that $|\Gamma(z(j), b_j)| = |\Gamma(y(j), b_j)| + 1$. By the Remark 4.2, this is equal to the position of b_j in $y(j)$. In other words, the position of b_j in $\widetilde{y(j)}$ is $\alpha_s = |\Gamma(z(j), b_j)|$.

On the other hand, $|\Gamma(z(j), b_j)|$ is equal to the number of entries of $z(j)$ which are larger than b_j . Therefore, in $c_{\alpha_s} > b_{\alpha_s} = b_j$. But this is a contradiction to $z(j) \leq_c y(j)$. Therefore, the first case, $c_j < a_i$ is not possible.

We assume that $a_i < c_j$. Since $a_j = b_j$, and since by our initial assumption $a_j < c_j$, we have that $b_j < c_j$. Since $i < j$, and since $\ell(y) = \ell(x) + 1$, Lemma 3.1 implies that $b_i \leq c_j$.

Assume for a second that $b_i < c_j$. Let α_s be the position of c_j in $\widetilde{z(j)}$. Since,

$$\{b_1, \dots, b_j\} \setminus \{b_i, b_j\} = \{c_1, \dots, c_j\} \setminus \{c_j, c_i\},$$

and since, $c_i < c_j$, $b_i < c_j$, and $b_j < c_j$, we see that $|\Gamma(z(j), c_j)| = |\Gamma(y(j), c_j)|$. Therefore, $b_{\alpha_s} < c_{\alpha_s} = c_i$. But this contradicts the fact that $z(j) \leq_c y(j)$.

Therefore, we assume that $b_i = c_j$. Since $b_j = a_j < c_j = b_i$, and since $\ell(y) = \ell(x) + 1$, Lemma 3.1 implies that $b_j \leq c_i = a_i < c_j$. We look at the position α_s of c_i in $\widetilde{z(j)}$. Since,

$$\{b_1, \dots, b_j\} \setminus \{b_i, b_j\} = \{c_1, \dots, c_j\} \setminus \{c_j, c_i\},$$

we see that $|\Gamma(z(j), c_i)| = |\Gamma(y(j), c_i)|$. Therefore, $b_{\alpha_s} < c_{\alpha_s} = c_i$. This contradicts the fact that $z(j) \leq_c y(j)$. We have handled all the cases, and the proof is complete. \square

Lemma 5.2. *Let $x = (a_1, \dots, a_n)$, $y = (b_1, \dots, b_n)$ and $z = (c_1, \dots, c_n)$ be three elements from R_n such that $a_k = b_k$ for all $k \in \{1, \dots, \widehat{i}, \dots, n\}$ and $a_i < b_i$. Furthermore, $c_k = b_k$ for $k = 1, \dots, i$. If $x \leq_D z \leq_D y$ and $\ell(y) = \ell(x) + 1$, then $z = y$.*

Proof. We proceed as in the proof of Lemma 5.1. Assume otherwise that $z \neq y$, and let $j > i$ be the first position where z differs from y . Hence, there are now two subcases; either $c_j < b_j$ or else $b_j < c_j$.

In the second case, with $b_j < c_j$, we see that $y(j) <_c z(j)$, which contradicts the fact that $z \leq_D y$.

Therefore, we assume that $c_j < b_j = a_j$. There are now two subcases; either $c_j < a_i$, or else $a_i < c_j$. We first treat the case $c_j < a_i$.

Recall that $\Gamma(z(j), c_j) = \{c_k \mid c_j < c_k, k = 1, \dots, j\}$, and that $\Gamma(x(j), c_j) = \{a_k \mid c_j < a_k, k = 1, \dots, j\}$. Then, since

$$\{a_1, \dots, a_j\} \setminus \{a_i, a_j\} = \{c_1, \dots, c_j\} \setminus \{c_j, c_i\},$$

and $c_j < a_i, a_j$, we see that $|\Gamma(z(j), c_j)| + 1 = |\Gamma(x(j), c_j)|$. This shows the following; if the position of c_j in $\widetilde{z(j)}$ is α_s , then $a_{\alpha_s} > c_{\alpha_s} = c_j$, a contradiction to $x(j) \leq_c z(j)$.

We proceed with the case that $a_i < c_j$. Since $\ell(y) = \ell(x) + 1$, and $z(j-1) = y(j-1)$, we see that c_j must be larger than $c_i = b_i = a_i + s + 1$ (or larger than $c_i = b_i = a_i + 1$). Therefore, similar to the above, since

$$\{a_1, \dots, a_n\} \setminus \{a_i, a_j\} = \{c_1, \dots, c_n\} \setminus \{c_j, c_i\},$$

and $a_i < c_j < a_j$, and $c_i < c_j$, we see that $|\Gamma(z(j), c_j)| + 1 = |\Gamma(x(j), c_j)|$. This shows the following; if the position of c_j in $\widetilde{z(j)}$ is α_s , then $a_{\alpha_s} > c_{\alpha_s} = c_j$, a contradiction to $x(j) \leq_c z(j)$.

Therefore, we conclude that $z = y$. \square

Lemma 5.3. *Let $x = (a_1, \dots, a_n)$ and $z = (c_1, \dots, c_n)$ be two elements from R_n . Suppose that $c_i = a_r$ and $c_r = a_i$, with $i < r$. Furthermore, suppose that $c_k = a_k$, for $k \notin \{i, r\}$. If $a_r > a_i$, then $z \not\leq_D x$.*

Proof. This follows directly from Corollary 4.4. \square

Proposition 5.4. *Let $x = (a_1, \dots, a_n)$ and $y = (b_1, \dots, b_n)$ be two elements from R_n such that $a_k = b_k$ for all $k \in \{1, \dots, \widehat{i}, \dots, n\}$ and $a_i < b_i$. Then $\ell(y) = \ell(x) + 1$ if and only if $y \rightarrow_D x$.*

Proof. It is clear from the hypotheses that $x <_{PPR} y$, and that $x <_D y$. We first show that if $\ell(y) = \ell(x) + 1$, then $y \rightarrow_D x$. Let $z = (c_1, \dots, c_n) \in R_n$ be such that $x \leq_D z \leq_D y$. Then, since $a_k = b_k$ for $k = 1, \dots, i-1$, we must have $c_k = a_k$, for $k = 1, \dots, i-1$. In other words, $x(k) = z(k) = y(k)$ for $k = 1, \dots, i-1$. Since $x(i) \leq_c z(i) \leq_c y(i)$, we must also

have $a_i \leq c_i \leq b_i$. Therefore, either $a_i = c_i$, or $a_i < c_i$. In the former case, by the Lemma 5.1, z is identically equal to x . Therefore, we have $a_i < c_i \leq b_i$, so that $x <_D z \leq_D y$. We are going to show that $z = y$.

As in the notation of Lemma 3.1, if $b_i = a_i + s + 1$ for some $s \geq 0$, then we must have $c_i = b_i$. This is because, c_i cannot be strictly larger than b_i (otherwise $z(i) > y(i)$), and c_i cannot be less than b_i (otherwise c_i has to be one of $\{a_{j_1}, \dots, a_{j_s}\}$, which contradicts with the fact that $z(k) = y(k)$ for all $k = 1, \dots, i-1$). Therefore, $c_k = b_k$ for $k = 1, \dots, i$. By the Lemma 5.2, we see that $z = y$. Therefore, $\ell(y) = \ell(x) + 1$ implies that $y \rightarrow_D x$.

Conversely, assume that $y \rightarrow_D x$. If $b_i = a_i + 1$, then it is clear that $\ell(y) = \ell(x) + 1$. So, we assume that $b_i = a_i + s + 1$, for some $s > 0$. To finish the proof, by the Lemma 3.1, it is enough to show that there exists a sequence of indices $1 \leq j_1 < \dots < j_s < i$ such that $\{a_{j_1}, \dots, a_{j_s}\} = \{a_i + 1, \dots, a_i + s\}$, and $b_i = a_i + s + 1$.

Let d be a number such that $1 \leq d \leq s$. If $a_i + d$ does not appear in y , then we define $z = (c_1, \dots, c_n) \in R_n$ to be the sequence such that $c_k = a_k$ for $k \in \{1, \dots, \hat{i}, \dots, n\}$ and $c_i = a_i + d$. It is clear that $x \leq_D z \leq_D y$. But this contradicts with the hypotheses that $y \rightarrow_D x$. Therefore, the number $a_i + d$ is an entry of y . Assume for a second that $a_i + d = b_t = a_t$ for some $t > i$. Then we define $z = (c_1, \dots, c_n) \in R_n$ to be the element such that $c_k = a_k$ for $k \in \{1, \dots, \hat{i}, \dots, \hat{t}, \dots, n\}$ and $c_i = a_i + d$, $c_t = a_i$. Then, using the Lemma 5.3, it is easy to check that $x \leq_D z \leq_D y$, which is a contradiction. Therefore, $t < i$. In other words, for any $1 \leq d < s$, the number $a_i + d$ is an entry of x , with the index $< i$. This shows that there exists a sequence of indices $1 \leq j_1 < \dots < j_s < i$ such that the set $\{a_{j_1}, \dots, a_{j_s}\}$ is equal to $\{a_i + 1, \dots, a_i + s\}$, and $b_i = a_i + s + 1$. \square

Lemma 5.5. *Let $x = (a_1, \dots, a_n)$, $y = (b_1, \dots, b_n)$ and $z = (c_1, \dots, c_n)$ be three element of R_n , such that $\tilde{x} = \tilde{y}$. If $x \leq_D z \leq_D y$, then $\tilde{z} = \tilde{x} = \tilde{y}$.*

Proof. By definition of the Deodhar ordering, $x \leq_D z \leq_D y$ is true if and only if $x(k) \leq_c z(k) \leq_c y(k)$, for all $k = 1, \dots, n$. Recall that \tilde{z} stands for the reordering, from the largest to smallest entries of z . Therefore, if $\tilde{z} \neq \tilde{x}$, then there exists $1 \leq \alpha_r \leq n$ such that $a_{\alpha_r} < c_{\alpha_r}$. But since $z(n) \leq_c y(n)$, we see that $c_{\alpha_r} \leq b_{\alpha_r} = a_{\alpha_r}$, a contradiction. Therefore $\tilde{z} = \tilde{x}$. \square

Lemma 5.6. *Let $x = (\underbrace{a_1, \dots, a_n}_{\sim})$, $y = (\underbrace{b_1, \dots, b_n}_{\sim})$ and $z = (c_1, \dots, c_n)$ be three elements from R_n such that $x(n-1) = y(n-1) = z(n-1)$, $a_n = b_n$ and $x \leq_D z \leq_D y$. Then, $c_n = a_n = b_n$.*

Proof. Since $\underbrace{x(n-1)}_{\sim} = \underbrace{y(n-1)}_{\sim}$, and since $\underbrace{a_n}_{\sim} = \underbrace{b_n}_{\sim}$, we see, by the Lemma 5.5, that $\tilde{z} = \tilde{x} = \tilde{y}$. This, together with the fact that $z(n-1) = x(n-1) = y(n-1)$, forces the equality $c_n = a_n = b_n$. \square

Proposition 5.7. *Let $x = (a_1, \dots, a_n)$ and $y = (b_1, \dots, b_n)$ be two elements of R_n . Suppose that for some $1 \leq i < j \leq n$, $a_j = b_i$, $a_i = b_j$ and $b_j < b_i$, and $a_k = b_k$ for all $k \in \{1, \dots, \widehat{i}, \dots, \widehat{j}, \dots, n\}$. Then, $\ell(y) = \ell(x) + 1$ if and only if $y \rightarrow_D x$.*

Proof. It is clear from Lemma 5.3 that $x <_D y$. Also, we know from Lemma 3.3 that $\ell(y) = \ell(x) + 1$ if and only if for each $s \in \{i + 1, \dots, j - 1\}$, either $a_j < a_s$, or $a_s < a_i$. Throughout the proof, we shall make use of this.

Suppose first that $y \rightarrow_D x$. Assume that there exists $s \in \{i + 1, \dots, j - 1\}$ such that $a_i < a_s < a_j$. Then, define $z = (c_1, \dots, c_n) \in R_n$ such that $c_k = a_k$ for all $k \notin \{s, j\}$, and, $c_s = a_j$, $c_j = a_s$. Then, by the repeated applications of Lemma 5.3, it is easy to see that $x \not\leq_D z \not\leq_D y$. But this implies that y does not cover x in the Deodhar ordering, which is a contradiction. Therefore, $\ell(y) = \ell(x) + 1$.

Conversely, suppose that $\ell(y) = \ell(x) + 1$. There are two cases; $j = i + 1$, or $j > i + 1$. Suppose first that $j = i + 1$. Notice that by the Lemma 5.5, the set of the entries of z is equal to the set of entries of x , which is also equal to the set of entries of y . Clearly, for $k = 1, \dots, i - 1$, we have that $x(k) = z(k) = y(k)$. Since $j = i + 1$, we see that $\widetilde{x(j)} = \widetilde{y(j)}$. Thus, by Lemma 5.5, we see that $\widetilde{z(j)} = \widetilde{x(j)} = \widetilde{y(j)}$. This shows that either $c_i = a_i$ and $c_j = a_j$, or $c_i = b_i$ and $c_j = b_j$. Finally, for $k > j$, Lemma 5.6 shows that $c_k = a_k = b_k$. Therefore, we conclude, in the case of $j = i + 1$, that either $z = x$, or $z = y$.

We proceed with the case that $j > i + 1$. By Lemma 3.3, we know that for $s = i + 1, \dots, j - 1$, either $a_j < a_s$, or $a_s < a_i$. Let $z = (c_1, \dots, c_n) \in R_n$ be such that $x \leq_D z \leq_D y$. Notice that by Lemma 5.5, the set of the entries of z is equal to the set of entries of x . Furthermore, for $k = 1, \dots, i - 1$, we have that $x(k) = z(k) = y(k)$. Also, since $x(i) \leq_c z(i) \leq_c y(i)$, we must have $a_i \leq c_i \leq b_i$. We proceed to show that for $s = i + 1, \dots, j - 1, j + 1, \dots, n$, $c_s = a_s = b_s$. Once we show this, the proof is finished as follows. By Lemma 5.5, we know that $\widetilde{z} = \widetilde{x} = \widetilde{y}$. Since $c_s = a_s = b_s$ for all $s \in \{1, \dots, \widehat{i}, \dots, \widehat{j}, \dots, n\}$, we either have $c_i = a_i$ and $c_j = a_j$, or $c_i = b_i$ and $c_j = b_j$, in other words, either $z = x$, or $z = y$.

We start by showing that $c_{i+1} = a_{i+1} = b_{i+1}$. By Lemma 3.3, we know that one of the following is true.

Case 1. $b_{i+1} = a_{i+1} < a_i$, or

Case 2. $b_{i+1} = a_{i+1} > b_i = a_j$.

We start with the first case that $a_{i+1} < a_i \leq c_i$, and we look at the following two subcases: $c_{i+1} < a_{i+1}$ or $c_{i+1} > a_{i+1}$.

Case 1.1. $c_{i+1} < a_{i+1} = b_{i+1}$, or

Case 1.2 $c_{i+1} > a_{i+1} = b_{i+1}$.

We first deal with the *Case 1.1.* Let $\Gamma(x(i + 1), c_{i+1}) = \{a_k \mid c_{i+1} < a_k, k = 1, \dots, i + 1\}$, and let $\Gamma(z(i + 1), c_{i+1}) = \{c_k \mid c_{i+1} < c_k, k = 1, \dots, i + 1\}$. Since

$$\{a_1, \dots, a_{i+1}\} \setminus \{a_i, a_{i+1}\} = \{c_1, \dots, c_{i+1}\} \setminus \{c_i, c_{i+1}\},$$

if $c_{i+1} < \widetilde{a_{i+1}}$, then $|\Gamma(x(i+1), c_{i+1})| = |\Gamma(z(i+1), c_{i+1})| + 1$. Hence, if the position of c_{i+1} in $\widetilde{z(i+1)}$ is c_{α_s} , then $a_{\alpha_s} > c_{\alpha_s}$. This is a contradiction with $x(i+1) \leq_c z(i+1)$.

Case 1.2. is similar; if $c_{i+1} > a_{i+1} = b_{i+1}$, then let $\Gamma(y(i+1), b_{i+1}) = \{b_k \mid b_{i+1} < b_k, k = 1, \dots, i+1\}$ and $\Gamma(z(i+1), b_{i+1}) = \{c_k \mid b_{i+1} < c_k, k = 1, \dots, i+1\}$. Since

$$\{b_1, \dots, b_{i+1}\} \setminus \{b_i, b_{i+1}\} = \{c_1, \dots, c_{i+1}\} \setminus \{c_i, c_{i+1}\},$$

$|\Gamma(z(i+1), b_{i+1})| = |\Gamma(y(i+1), b_{i+1})| + 1$. Therefore, if the position of b_{i+1} in $\widetilde{y(i+1)}$ is $b_{\alpha_{s'}}$, then $c_{\alpha_{s'}} > b_{\alpha_{s'}}$. This is a contradiction with $z(i+1) \leq_c y(i+1)$.

We proceed with *Case 2.* that $b_{i+1} = a_{i+1} > b_i = a_j$. Once again, there are two subcases;

Case 2.1. $c_{i+1} < a_{i+1} = b_{i+1}$, or

Case 2.2. $c_{i+1} > a_{i+1} = b_{i+1}$.

We continue with *Case 2.1.* Since,

$$\{a_1, \dots, a_{i+1}\} \setminus \{a_i, a_{i+1}\} = \{c_1, \dots, c_{i+1}\} \setminus \{c_i, c_{i+1}\}.$$

we have that $|\Gamma(x(i+1), a_{i+1})| \geq |\Gamma(z(i+1), a_{i+1})| + 1$. So, if the position of a_{i+1} in $\widetilde{x(i+1)}$ is a_{α_s} , then $a_{\alpha_s} > c_{\alpha_s}$. This is a contradiction with $x(i+1) \leq_c z(i+1)$.

Finally, we look at *Case 2.2.* Since

$$\{b_1, \dots, b_{i+1}\} \setminus \{b_i, b_{i+1}\} = \{c_1, \dots, c_{i+1}\} \setminus \{c_i, c_{i+1}\},$$

and since, $c_i \leq b_i < b_{i+1}$ we see that $|\Gamma(z(i+1), b_{i+1})| = |\Gamma(y(i+1), b_{i+1})| + 1$. Therefore, if the position of b_{i+1} in $\widetilde{y(i+1)}$ is $b_{\alpha_{s'}}$, then $c_{\alpha_{s'}} > b_{\alpha_{s'}}$. This is a contradiction with $z(i+1) \leq_c y(i+1)$.

We have dealt with all of the cases. We conclude that $c_{i+1} = a_{i+1} = b_{i+1}$. Notice that, as long as $a_k = b_k$ and $i < k < j$, the same arguments above work. Therefore, for any $k = i+1, \dots, j-1$ we have $c_k = a_k = b_k$.

Note also that $\widetilde{x(j)} = \widetilde{y(j)}$. By Remark 4.6, we know that $x(j) \leq_D z(j) \leq_D y(j)$. Hence, by Lemma 5.5, $x(j) = y(j) = z(j)$. Since $c_k = a_k = b_k$ for $k \notin \{i, j\}$, we either have that $c_i = a_i$, $c_j = a_j$, or that $c_i = a_j$, $c_j = a_i$. Therefore, we either have that $z(j) = y(j)$, or that $z(j) = x(j)$.

Finally, for $k > j$, Lemma 5.6 shows that $c_k = a_k = b_k$. This shows that $z = y$ or $z = x$, hence y covers x , and hence the proof is complete. \square

Remark 5.8. Propositions 5.4 and 5.7 show that a covering for the Pennell-Putcha-Renner ordering is a covering for the Deodhar ordering. Proposition 5.11 below shows that the converse is also true.

Lemma 5.9. Let $x = (a_1, \dots, a_n), y = (b_1, \dots, b_n) \in R_n$. Suppose that there exists $i \in \{1, \dots, n-1\}$ such that

- (1) $a_k = b_k$ for $k = 1, \dots, i-1$, and $b_i > a_i$,

(2) $b_i = a_r$ for some $r > i$.

Then, $y \rightarrow_D x$ implies that $y \rightarrow_{PPR} x$.

Proof. Our strategy for proving that $y \rightarrow_D x$ implies $y \rightarrow_{PPR} x$ is as follows. We construct an element $z \in R_n$, such that $x \not\leq_D z \leq_D y$ and the pair $x, z \in R_n$ satisfy the hypothesis of the Proposition 5.7. Thus, $z \rightarrow_D x$ implies that $\ell(z) = \ell(x) + 1$, and this, by Lemma 3.3 this implies that $z \rightarrow_{PPR} x$. First, assume that $a_i = 0$. Let r' be the smallest index such that $i < r' \leq r$, and $a_{r'}$ is nonzero. Define $z = (c_1, \dots, c_n)$ by setting $c_k = a_k$ if $k \notin \{i, r'\}$, and $c_i = a_{r'}$, $c_{r'} = a_i$. It is easy to check that (see the proof of case $a_i > 0$, below) $x \not\leq_D z \leq_D y$, and that the pair x, z satisfy the hypothesis of Proposition 5.7. Therefore, we are done in the case that $a_i = 0$. We proceed with the assumption that $a_i > 0$.

Let r' be the smallest integer such that

- (1) $i < r' \leq r$,
- (2) $a_i < a_{r'}$.

Therefore,

$$(5.1) \quad \text{if } i < s < r', \text{ then } a_s < a_i.$$

We define $z = (c_1, \dots, c_n) \in R_n$ as follows. Let $k \in \{1, \dots, \widehat{i}, \dots, \widehat{r'}, \dots, n\}$. Set $c_k = a_k$. Also, set $c_i = a_{r'}$, and $c_{r'} = a_i$. It is easy to check that $x \not\leq_D z$. We are going to show that $z \leq_D y$. Note the following

- (1) $\underline{x(k)} = \underline{y(k)} = \underline{z(k)}$ for $k = 1, \dots, i-1$.
- (2) $\underline{x(i)} \leq_c \underline{z(i)} \leq_c \underline{y(i)}$.
- (3) $\underline{z(k)} = \underline{x(k)} \leq_c \underline{y(k)}$ for $k = r', \dots, n$.

Therefore, it is enough to prove that $\underline{z(k)} \leq_c \underline{y(k)}$ for $k = i+1, \dots, r'-1$. To this end, $k \in \{i+1, \dots, r'-1\}$, and let $1 \leq m \leq k$. We are going to show that $|\Gamma(z(k), c_m)| \leq |\Gamma(y(k), c_m)|$.

There are two cases; $c_m < a_i$, or $c_m \geq a_i$. We start with the first one.

Since $c_m < a_i$, $m \notin \{i, r\}$, hence $a_m = c_m$. The set of entries of $\underline{z(k)}$ that are larger than $c_m = a_m$ is equal to the set of entries of $\underline{x(k)}$ which are larger than a_m . Therefore,

$$(5.2) \quad |\Gamma(z(k), c_m)| = |\Gamma(x(k), c_m)| \leq |\Gamma(y(k), c_m)|, \text{ if } c_m < a_i.$$

The next case we check is that $c_m \geq a_i = c_{r'}$. By the observation (5.1) above,

$$(5.3) \quad |\Gamma(z(k), c_m)| = |\Gamma(z(i), c_m)|.$$

On the other hand, since $\underline{z(i)} \leq_c \underline{y(i)}$,

$$|\Gamma(z(i), c_m)| \leq |\Gamma(y(i), c_m)|,$$

and since $i < k$, we have

$$|\Gamma(y(i), c_m)| \leq |\Gamma(y(k), c_m)|.$$

Therefore,

$$(5.4) \quad |\Gamma(z(k), c_m)| \leq |\Gamma(y(k), c_m)|, \text{ if } c_m \geq a_i.$$

Hence, (5.2) and (5.4) shows that $z(k) \leq_c y(k)$ for $k \leq r' - 1$. Having constructed $z \in R_n$, such that $x \leq_D z \leq_D y$, since y covers x (in the Deodhar ordering), we have that $z = y$. Thus, we are exactly as in the hypotheses of the Proposition 5.7. Therefore, we have that $\ell(y) = \ell(x) + 1$, and that $y \rightarrow_{PPR} x$. □

Lemma 5.10. *Let $x = (a_1, \dots, a_n), y = (b_1, \dots, b_n) \in R_n$. Suppose that there exists $i \in \{1, \dots, n-1\}$ such that*

- (1) $a_k = b_k$ for $k = 1, \dots, i-1$, and $b_i > a_i$,
- (2) $b_i \notin \{a_1, \dots, a_n\}$.

Then, $y \rightarrow_D x$ implies that $y \rightarrow_{PPR} x$.

Proof. We make use of the following set

$$\gamma(x, i) = \{a_t : t > i, a_i > a_t\}.$$

There are two cases; $\gamma(x, i) = \emptyset$, or $\gamma(x, i) \neq \emptyset$. We start with the first case that $\gamma(x, i) = \emptyset$.

Define $z = (c_1, \dots, c_n)$ as follows. Let $c_k = a_k$ for $k \neq i$, and let $c_i = b_i$. Clearly $x \leq_D z$. We are going to show that $z \leq_c y$.

It is enough to show that

$$|\Gamma(z(k), c_m)| \leq |\Gamma(y(k), c_m)|,$$

for $k > i$, and $1 \leq m \leq k$.

To this end, let $1 \leq m \leq k$, and $i < k$. If $c_m \geq a_i$, then

$$|\Gamma(z(k), c_m)| = |\Gamma(z(i), c_m)| = |\Gamma(y(i), c_m)| \leq |\Gamma(y(k), c_m)|.$$

If $c_m < a_i$, then $c_m = a_m$, and

$$|\Gamma(z(k), c_m)| = |\Gamma(x(k), a_m)| \leq |\Gamma(y(k), a_m)| = |\Gamma(y(k), c_m)|.$$

Therefore, if $\gamma(x, i) = \emptyset$, then $z \leq_D y$.

Having constructed $z \in R_n$, such that $x \leq_D z \leq_D y$, since y covers x (in the Deodhar ordering), we have that $z = y$. Thus, we are exactly as in the hypotheses of the Proposition 5.7. Therefore, we have that $\ell(y) = \ell(x) + 1$, and that $y \rightarrow_{PPR} x$.

We continue with the case where $\gamma(x, i) \neq \emptyset$. Once again, there are two subcases; either there exists $a_t \in \gamma(x, i)$ such that $b_i > a_t$, or for every $a_t \in \gamma(x, i)$, $a_t > b_i$.

We proceed with the first one. Then, there exists $a_t \in \gamma(x, i)$ such that $b_i > a_t$. Let t' be the smallest number such that

- (1) $i < t'$,
- (2) $a_i < a_{t'} < b_i$.

Therefore, if $i < s < t'$, then

$$(5.5) \quad a_i > a_s.$$

Define $z = (c_1, \dots, c_n)$ as follows. If $k \notin \{i, t'\}$, then $c_k = a_k$, and $c_i = a_{t'}$, $c_{t'} = a_i$. Clearly $x \leq_D z$. We are going to show that $z \leq_c y$. It is enough to show that

- (1) $\underline{x(k)} = \underline{y(k)} = \underline{z(k)}$ for $k = 1, \dots, i-1$.
- (2) $\underline{x(i)} \leq_c \underline{z(i)} \leq_c \underline{y(i)}$.
- (3) $\underline{z(k)} = \underline{x(k)} \leq_c \underline{y(k)}$ for $k = t', \dots, n$.

Therefore, it is enough to prove that $z(k) \leq_c y(k)$ for $k = i+1, \dots, t'-1$. To this end, $k \in \{i+1, \dots, t'-1\}$, and let $1 \leq m \leq k$. We are going to show that $|\Gamma(z(k), c_m)| \leq |\Gamma(y(k), c_m)|$.

There are two cases; $c_m < a_i$, or $c_m \geq a_i$. We start with the first one.

Since $c_m < a_i$, $m \notin \{i, t'\}$, hence $a_m = c_m$. The set of entries of $z(k)$ that are larger than $c_m = a_m$ is equal to the set of entries of $x(k)$ which are larger than a_m . Therefore,

$$(5.6) \quad |\Gamma(z(k), c_m)| = |\Gamma(x(k), c_m)| \leq |\Gamma(y(k), c_m)|, \text{ if } c_m < a_i.$$

To deal with the other case we check that $c_m \geq a_i = c_{t'}$. By the observation (5.5) above,

$$(5.7) \quad |\Gamma(z(k), c_m)| = |\Gamma(z(i), c_m)|.$$

On the other hand, since $z(i) \leq_c y(i)$,

$$|\Gamma(z(i), c_m)| \leq |\Gamma(y(i), c_m)|,$$

and since $i < k$, we have

$$|\Gamma(y(i), c_m)| \leq |\Gamma(y(k), c_m)|.$$

Therefore,

$$(5.8) \quad |\Gamma(z(k), c_m)| \leq |\Gamma(y(k), c_m)|, \text{ if } c_m \geq a_i.$$

Hence, (5.6) and (5.8) show that $z(k) \leq_c y(k)$ for $k \leq t'-1$.

We proceed with the case that $\gamma(x, i) \neq \emptyset$, and $a_t > b_i$, for all $a_t \in \gamma(x, i)$.

Define $z = (c_1, \dots, c_n)$ as follows. If $k \neq i$, then $c_k = a_k$, and $c_i = b_i$. Clearly $x \leq_D z$. We are going to show that $z \leq_c y$.

It is enough to show that

$$|\Gamma(z(k), c_m)| \leq |\Gamma(y(k), c_m)|,$$

for $k > i$, and $1 \leq m \leq k$.

To this end, let $1 \leq m \leq k$, and $i < k$. If $c_m \geq b_i$, then

$$|\Gamma(z(k), c_m)| = |\Gamma(x(k), c_m)| \leq |\Gamma(y(i), c_m)|.$$

If $c_m < b_i$, then $m < i$, and $c_m = a_m = b_m$. Note that the following. If $t > i$, then $b_t > b_i$. Assume otherwise. Let $i < t$ be the smallest number such that $b_i > b_t$. Then,

$$|\Gamma(y(t), b_i)| < |\Gamma(x(k), b_i)|,$$

which is a contradiction. Hence,

$$|\{c_s : i < s \leq k, c_s > b_i\}| = |\{b_s : i < s \leq k, b_s > b_i\}| = k - i + 1$$

Therefore,

$$\begin{aligned} |\Gamma(z(k), c_m)| &= |\{c_s : i \geq s, c_s > c_m\}| + |\{c_s : i < s \leq k, c_s > c_m\}| \\ &= |\{b_s : i \geq s, b_s > c_m\}| + |\{b_s : i < s \leq k, b_s > b_i\}| \\ &= |\{b_s : i \geq s, b_s > c_m\}| + |\{b_s : i < s \leq k, b_s > c_m\}| \\ &= |\Gamma(y(k), c_m)|. \end{aligned}$$

Therefore, if $\gamma(x, i) \neq \emptyset$, then $z \leq_D y$. Having constructed $z \in R_n$, such that $x \leq_D z \leq_D y$, since y covers x (in the Deodhar ordering), we have that $z = y$. Thus, we are exactly as in the hypotheses of the Proposition 5.7. Therefore, we have that $\ell(y) = \ell(x) + 1$, and that $y \rightarrow_{PPR} x$.

We have handled all the cases, and the proof is complete. \square

Proposition 5.11. *Let $x = (a_1, \dots, a_n)$ and $y = (b_1, \dots, b_n)$ be two elements from R_n . Suppose that $y \rightarrow_D x$. Then $y \rightarrow_{PPR} x$.*

Proof. Let $i \in \{1, \dots, n-1\}$ be the smallest index such that $k = 1, \dots, i-1, a_k = b_k$ and $b_i > a_i$.

Then we have either

Case 1. $b_i = a_r$ for some $r > i$, or

Case 2. $b_i \notin \{a_1, \dots, a_n\}$.

Then, in the *Case 1.*, the Lemma 5.9 shows that $y \rightarrow_{PPR} x$, and similarly, in the *Case 2.*, the Lemma 5.10 shows that $y \rightarrow_{PPR} x$. \square

Theorem 5.12. *The Deodhar ordering \leq_D on R_n is the same as Pennell-Putcha-Renner ordering \leq_{PPR} on R_n .*

Proof. By the Proposition 5.4, and the Proposition 5.7 we know that $y \rightarrow_{PPR} x$ implies $y \rightarrow_D x$. Conversely, by the Proposition 5.11, if $y \rightarrow_D x$, then $y \rightarrow_{PPR} x$. Therefore, the two orderings have the same covering relations, hence they are the same order. \square

Corollary 5.13. (*Deodhar*) Let $x = (a_1, \dots, a_n)$ and $y = (b_1, \dots, b_n)$ be two permutations. Then, $x \leq y$ in the Bruhat ordering \leq on S_n if and only if $x \leq_D y$ in the Deodhar ordering on S_n .

REFERENCES

- [1] E. Miller, B. Sturmfels, *Combinatorial commutative algebra*. Graduate Texts in Mathematics, 227. Springer-Verlag, New York, 2005.
- [2] E.A. Pennell M. Putcha, L. Renner, *Analogue of the Bruhat-Chevalley Order for Reductive Monoids*. Journal of Algebra **196** (1997), 339-368.
- [3] M. Putcha, *Shellability in Reductive Monoids*. Tran. Amer. Math. Soc. 354 (2001), 413-426.
- [4] L.E. Renner, *Analogue of the Bruhat decomposition for algebraic monoids*. Journal of Algebra **101** (1986), 303-338.
- [5] L.E. Renner, *Analogue of the Bruhat decomposition for algebraic monoids II: the length function and the trichotomy*. Journal of Algebra **188** (1997), 272-291.

UNIVERSITY OF WESTERN ONTARIO, CANADA

E-mail address: mcan@uwo.ca, lex@uwo.ca